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Lebesgue Constants for Certain Classes of Nodes

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For each $f \in C(I)$, let $B_n(f)$ be the best uniform polynomial approximation of degree at most n, and let $e_n(f) = f - B_n(f)$ be the error function. Denote the set of extreme points of $e_n(f)$ by $E_n(f)$, and assume that this set has precisely $n+2$ points. If $E(\hat{\mathbf{F}})$ is the infinite triangular array of nodes whose nth row consists of the $n+2$ points of $E_n(f)$, then the corresponding Lebesgue constant of order n is designated $\Lambda_n(E(\hat{\mathbf{F}}))$. For certain rational and non-rational functions it is shown that $\Lambda_n(E(\hat{\mathbf{F}})) = O(\log n)$.

1. INTRODUCTION

Let $-1 \leqslant x_0^n < x_1^n < \cdots < x_n^n < x_{n+1}^n \leqslant 1$ be $n+2$ points in the interval $I = [-1, 1]$. Setting

$$\mathbf{X}_n = \{x_i^n\}_{i=0}^{n+1}, \tag{1.1}$$

$$\mathbf{X} = \{\mathbf{X}_n\}_{n=0}^{\infty} \tag{1.2}$$

is an infinite triangular array of nodes [11, p. 88]. Let

$$\{l_i^{(n)}(x)\}_{i=0}^{n+1} \tag{1.3}$$

be the fundamental Lagrange polynomials determined by (1.1), [11, p. 88]. The Lebesgue function of order $n+1$ determined by $\mathbf{X}$ is then

$$\lambda_{n+1}(\mathbf{X}, x) = \sum_{i=0}^{n+1} |l_i^{(n)}(x)|, \tag{1.4}$$

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and the Lebesgue constant of order $n+1$ determined by $\mathbf{X}$ is defined [11, p. 89] to be

$$\Lambda_{n+1}(\mathbf{X}) = \max_{-1 \leqslant x \leqslant 1} \lambda_{n+1}(\mathbf{X}, x). \tag{1.5}$$

A classical problem of approximation theory is to estimate $\Lambda_{n+1}(\mathbf{X})$ as a function of n and $\mathbf{X}$.

Let C_{n+1} be the Chebyshev polynomial of degree $n+1$. If $g_n(x) = x^{n+1}$, and $B_n(g_n)$ is the best uniform polynomial approximation of degree $\leqslant n$ to g_n on I, then it is well known that the error function

$$e_n(g_n)(x) = x^{n+1} - B_n(g_n)(x), \qquad x \in I,$$

satisfies

$$e_n(g_n)(x) = 1/2^n\, C_{n+1}(x), \qquad x \in I. \tag{1.6}$$

The set of extreme points, $E_n(g_n)$, of the error function is defined by

$$E_n(g_n) = \{x \in I: |e_n(g_n)(x)| = \|e_n(g_n)\|\}, \quad \|\ \| = \max_{x \in I} |\ |.$$

Thus

$$\mathbf{G} = \{E_n(g_n)\}_{n=0}^{\infty} \tag{1.7}$$

is the infinite triangular array of nodes whose nth row consists of the $n+2$ extreme points of C_{n+1}. It is known [1, 4] that the Lebesgue constants determined by $\mathbf{G}$ satisfy

$$\Lambda_{n+1}(\mathbf{G}) = 0(\log(n+1)). \tag{1.8}$$

Although $\Lambda_{n+1}(\mathbf{G})$ does not equal [3, 9]

$$\min_{\mathbf{X}} \Lambda_{n+1}(\mathbf{X}),$$

the order of magnitude displayed by $\Lambda_{n+1}(\mathbf{G})$ in (1.8) is optimal in the sense that there exists a positive constant α, independent of n, such that

$$0 < \alpha < \Lambda_{n+1}(\mathbf{X})/\log(n+1) \tag{1.9}$$

for every infinite triangular array of nodes of the type (1.2), [5, 9, 11].

There are other infinite triangular arrays of nodes $\mathbf{X}$ with Lebesgue constants satisfying

$$\Lambda_{n+1}(\mathbf{X}) = 0(\log(n+1)); \tag{1.10}$$

see, for example, [14]. Perhaps the most familar infinite triangular array of nodes with a Lebesgue constant satisfying (1.10) is the array T whose nth row (using the convention that the nth row contains $n+2$ elements) consists of the zeros of C_{n+2}, [1, 4, 11].

The main objective of the present paper is to demonstrate a new class of infinite triangular arrays of nodes with Lebesgue constants satisfying (1.10). As was the case in (1.7), the origins of these arrays will be best approximation problems.

2. Preliminaries

Let $C(I)$ denote the space of real-valued, continuous functions on the interval $I=[-1,1]$, and let $\pi_n \subseteq C(I)$ be the space of real polynomials of degree at most n. Denote, as above, the uniform norm on $C(I)$ by $\|\cdot\|$. For each $f \in C(I)$ with best approximation $B_n(f)$ from π_n, let

$$e_n(f)(x)=f(x)-B_n(f)(x), \qquad x \in I. \tag{2.1}$$

Then the set of extreme points $E_n(f)$ of $e_n(f)$ is given by

$$E_n(f)=\{x \in I: |e_n(f)(x)|=\|e_n(f)\|\}. \tag{2.2}$$

It is well known [2] that $E_n(f)$ contains at least $n+2$ points. Let

$$\hat{\mathbf{F}}=\{f_n\}_{n=1}^{\infty} \subseteq C(I). \tag{2.3}$$

Suppose, for each n, that $E_n(f_n)$ contains precisely $n+2$ points. Then the class

$$E(\hat{\mathbf{F}})=\{E_n(f_n)\}_{n=1}^{\infty} \tag{2.4}$$

forms an infinite triangular array of nodes of the type given in (1.2). Therefore (2.4) determines, for each n, a Lebesgue constant

$$\Lambda_{n+1}(E(\hat{\mathbf{F}})). \tag{2.5}$$

The remainder of the paper will focus on the Lebesgue constants generated by a certain class of rational functions in the manner prescribed by (2.3), (2.4), and (2.5), and on Lebesgue constants generated by

$$E_f=\{E_n(f)\}_{n=1}^{\infty}, \tag{2.6}$$

for certain functions $f \in C(I)$. We note that (2.4) yields (2.6) whenever $\hat{\mathbf{F}}$ in (2.3) is a singleton; that is, when $f_n=f$, $n=1,2,\ldots$.

The functions and corresponding infinite triangular arrays of nodes to be subsequently analyzed will result in Lebesgue functions of optimal order.

3. Rational Functions

Suppose that

$$\{a_n\}_{n=1}^{\infty} \tag{3.1}$$

is a (possibly unbounded) sequence of numbers satisfying

$$a_n \geqslant 2, \qquad n = 1, 2, \ldots. \tag{3.2}$$

Let

$$r_n(x) = 1/(a_n - x), \qquad x \in I. \tag{3.3}$$

Then $r_n^{(n+1)}(x) > 0$ for $x \in I$, and consequently

$$E_n(r_n) = \{x \in I : |e_n(r_n)(x)| = \|e_n(r_n)\|\} \tag{3.4}$$

contains precisely $n + 2$ points. If

$$\mathbf{R} = \{r_n\}_{n=1}^{\infty}, \tag{3.5}$$

then

$$E(\mathbf{R}) = \{E_n(r_n)\}_{n=1}^{\infty} \tag{3.6}$$

determines the Lebesgue constant

$$\Lambda_{n+1}(E(\mathbf{R})). \tag{3.7}$$

The principal result of Section 3 is the following theorem.

THEOREM 1. *Let $\{a_n\}_{n=1}^{\infty}$ satisfy (3.2), and let r_n be defined as in (3.3). If $\mathbf{R} = \{r_n\}_{n=1}^{\infty}$, then the Lebesgue constant defined in (3.7) satisfies*

$$\Lambda_{n+1}(E(\mathbf{R})) = 0(\log(n+1)). \tag{3.8}$$

Prior to effecting the proof of Theorem 1, the statements of two lemmas are needed.

LEMMA 1. *Let*

$$E_n(r_n) = \{t_0^n, t_1^n, \ldots, t_n^n, t_{n+1}^n\}, \tag{3.9}$$

where

$$-1 = t_0^n < t_1^n < \cdots < t_n^n < t_{n+1}^n = 1. \tag{3.10}$$

Let

$$z_k^n = \cos \frac{(n+1-k)}{n+1} \pi, \qquad k = 0, 1, \ldots, n+1,$$

and

$$\zeta_k^n = \cos \frac{(n-k)}{n} \pi, \qquad k = 0, 1, \ldots, n. \tag{3.11}$$

Then

$$z_k^n < t_k^n < \zeta_k^n, \qquad k = 1, \ldots, n. \tag{3.12}$$

We note that $\{z_k^n\}_{k=0}^{n+1}$ are the extreme points of C_{n+1}, and that $\{\zeta_k^n\}_{k=0}^{n}$ are the extreme points of C_n. Lemma 1 is an immediate consequence of [13, Theorem 3.3]. The superscript notation employed in Lemma 1 was used to emphasize the dependence of $E_n(r_n)$ on n. Hereafter this dependence is assumed, and consequently, except in cases of emphasis, the superscripts are omitted.

LEMMA 2. *Let* $E_n(r_n) = \{t_i\}_{i=0}^{n+1}$ *be the extreme set defined in* (3.4) *and* (3.9). *Define* w_n *by*

$$w_n(x) = \prod_{j=0}^{n+1} (x - t_j). \tag{3.13}$$

Then

$$0 < C_1 \leqslant \|w_n'\| / |w_n'(t_i)| \leqslant C_2 < +\infty, \qquad i = 0, 1, \ldots, n+1, \tag{3.14}$$

and

$$0 < C_3 \leqslant |w_n'(t_i)| / |w_n'(t_{i+1})| \leqslant C_4 < +\infty, \qquad i = 0, 1, \ldots, n, \tag{3.15}$$

where C_1, C_2, C_3, *and* C_4 *are positive constants that are independent of* n.

Lemma 2 is an immediate consequence of expressions (2.26) and (2.28)–(2.31) ($a = a_n$) in [6].

Proof of Theorem 1. Let $x \in I$, $x \neq t_j$, $j = 0, 1, \ldots, n+1$. As in Lemma 1, let $E_n(r_n) = \{t_0, t_1, \ldots, t_n, t_{n+1}\}$, with ordering (3.10). Suppose that $t_k < x < t_{k+1}$, where $0 \leqslant k \leqslant n$. Then (3.14) implies that

$$\left| \frac{w_n(x)}{(x - t_i)\, w_n'(t_i)} \right| \leqslant C_2, \qquad i = 0, 1, \ldots, n+1, \tag{3.16}$$

where C_2 is independent of n. Thus from (1.4), (3.13), and (3.16),

$$\begin{aligned}\lambda_{n+1}(E(\mathbf{R}), x) &= \sum_{i=0}^{n+1} \left|\frac{w_n(x)}{(x-t_i)\, w_n'(t_i)}\right| \\ &\leqslant 5C_2 + \sum_{i=1}^{k-2} \left|\frac{w_n(x)}{(x-t_i)\, w_n'(t_i)}\right| \\ &\quad + \sum_{i=k+2}^{n} \left|\frac{w_n(x)}{(x-t_i)\, w_n'(t_i)}\right|, \end{aligned} \tag{3.17}$$

where as usual $\sum_{i=r}^{s} \theta_i = 0$ if $r > s$. Let

$$I = \sum_{i=1}^{k-2} \left|\frac{w_n(x)}{(x-t_i)\, w_n'(t_i)}\right|, \tag{3.18}$$

and

$$\bar{I} = \sum_{i=k+2}^{n} \left|\frac{w_n(x)}{(x-t_i)\, w_n'(t_i)}\right|. \tag{3.19}$$

Now equalities (2.24) and (2.28) of [6] imply that

$$\begin{aligned} I &\leqslant \frac{|(1-x^2)[n(a_n^2-1)^{1/2} C_n(x) + (a_n x - 1)\, C_n'(x)]|}{n\,|(a_n-1)\, n + (a_n^2-1)^{1/2}|} \sum_{i=1}^{k-2} \frac{1}{|x-t_i|} \\ &\leqslant \frac{(a_n^2-1)^{1/2}(1-x^2)|C_n(x)|}{\beta_n} \sum_{i=1}^{k-2} \frac{1}{|x-t_i|} \\ &\quad + \frac{(a_n+1)(1-x^2)|C_n'(x)|}{n\beta_n} \sum_{i=1}^{k-2} \frac{1}{|x-t_i|}, \end{aligned} \tag{3.20}$$

where

$$\beta_n = (a_n - 1)\, n + (a_n^2 - 1)^{1/2}. \tag{3.21}$$

Let I_1 be the first term on the right in (3.20), and let I_2 be the second term.

We first show that

$$I_2 = 0(\log(n+1)). \tag{3.22}$$

Since $t_k < x < t_{k+1}$, (3.12) implies that

$$|x - t_i| > |x - \zeta_i|, \qquad i = 1, \ldots, k-2, \tag{3.23}$$

where $\{\zeta_i\}_{i=0}^{n}$ are the extreme points (3.11) of C_n. From (3.20) and (3.23),

$$
\begin{aligned}
I_2 &= \frac{(a_n+1)}{n\beta_n} \sum_{i=1}^{k-2} \frac{|(x^2-1)\,C_n'(x)|}{|x-t_i|} \\
&\leqslant \frac{(a_n+1)}{n\beta_n} \sum_{i=1}^{k-2} \frac{|(x^2-1)\,C_n'(x)|}{|x-\zeta_i|} \\
&= \frac{n^2(a_n+1)}{n\beta_n} \sum_{i=1}^{k-2} \frac{|(x^2-1)\,C_n'(x)|}{|x-\zeta_i|\,|(1-\zeta_i^2)\,C_n''(\zeta_i)|}.
\end{aligned} \tag{3.24}
$$

Now (1.7), (1.8), (3.21), and (3.24) imply the validity of (3.22). Returning to (3.20),

$$
\begin{aligned}
I_1 &= \frac{(a_n^2-1)^{1/2}(1-x^2)|C_n(x)|}{\beta_n} \sum_{i=1}^{k-2} \frac{1}{|x-t_i|} \\
&\leqslant \frac{(a_n^2-1)^{1/2}(1-x^2)}{\beta_n} \left| \frac{C_{n+1}'(x)}{2(n+1)} - \frac{C_{n-1}'(x)}{2(n-1)} \right| \sum_{i=1}^{k-2} \frac{1}{|x-t_i|} \\
&\leqslant \frac{(a_n^2-1)^{1/2}}{2(n-1)\beta_n} \sum_{i=1}^{k-2} \frac{|(x^2-1)\,C_{n-1}'(x)|}{|x-t_i|} \\
&\quad + \frac{(a_n^2-1)^{1/2}}{2(n+1)\beta_n} \sum_{i=1}^{k-2} \frac{|(x^2-1)\,C_{n+1}'(x)|}{|x-t_i|}.
\end{aligned} \tag{3.25}
$$

As in (3.23), (3.12) implies that

$$
|x-t_i| > |x-z_{i+1}|, \qquad i=1,\dots,k-2. \tag{3.26}
$$

On the other hand, if $-1=y_0<y_1<\cdots<y_{n-2}<y_{n-1}=1$ are the extreme points of C_{n-1}, then (3.12) implies that

$$
\zeta_l < y_l, \qquad l=1,2,\dots,n-1. \tag{3.27}
$$

Thus (3.27) and (3.23) imply that

$$
|x-t_i| > |x-y_i|, \qquad i=1,2,\dots,k-2. \tag{3.28}
$$

Utilizing (3.26) and (3.28) in (3.25) results in

$$
\begin{aligned}
I_1 &\leqslant \frac{(a_n^2-1)^{1/2}(n+1)^2}{2(n+1)\beta_n} \sum_{i=1}^{k-2} \frac{|(x^2-1)\,C_{n+1}'(x)|}{|x-z_{i+1}|\,|(1-z_{i+1}^2)\,C_{n+1}''(z_{i+1})|} \\
&\quad + \frac{(a_n^2-1)^{1/2}(n-1)^2}{2(n-1)\beta_n} \sum_{i=1}^{k-2} \frac{|(x^2-1)\,C_{n-1}'(x)|}{|x-y_i|\,|(1-y_i^2)\,C_{n-1}''(y_i)|}.
\end{aligned}
$$

This inequality, (1.7), (1.8), and (3.21) now imply that

$$I_1 = 0(\log(n+1)). \tag{3.29}$$

Combining (3.22) and (3.29), we have that

$$I = 0(\log(n+1)) \tag{3.30}$$

whenever $t_k < x < t_{k+1}$, $0 \leqslant k \leqslant n$. The expression $\bar{I}$ given in (3.19) can be treated in a manner similar to that given for I, and consequently

$$\bar{I} = 0(\log(n+1)). \tag{3.31}$$

Thus (3.30) and (3.31) imply that

$$\lambda_{n+1}(E(\mathbf{R}), x) = 0(\log(n+1)), \qquad x \in I. \tag{3.32}$$

Equalities (1.4), (1.5), and (3.32) now imply the conclusion of Theorem 1. ∎

COROLLARY 1. *Let the nth row of the infinite triangular array of nodes* **A** *be given by*

$$\mathbf{A}_n = \{t_0^n, t_1^n, \ldots, t_{n+1}^n\},$$

where $t_0^n = -1$, $t_{n+1}^n = 1$, *and* $\{t_i^n\}_{i=1}^n$ *are the n zeros of*

$$n(a_n^2 - 1)^{1/2}\, C_n(x) + (a_n x - 1)\, C_n'(x) = 0. \tag{3.33}$$

Then $\Lambda_{n+1}(\mathbf{A}) = 0(\log(n+1))$.

Proof. Since $\mathbf{A} = E(\mathbf{R})$ [10, p. 35], the result is immediate. ∎

COROLLARY 2. *Let* $\alpha \geqslant \beta > 0$ *be constants not depending on n. Define*

$$U_n(x) = 1/(\alpha(n+2) + 2 - x), \qquad x \in I,$$

and

$$V_n(x) = 1/(\beta(n+2) - 2 - x), \qquad x \in I, \tag{3.34}$$

where n is large enough to ensure that the denominator of V_n *doesn't vanish on I. Let* $E_n(U_n)$ *and* $E_n(V_n)$ *consist of*

$$-1 = u_0 < u_1 < u_2 < \cdots < u_n < u_{n+1} = 1,$$

and

$$-1 = v_0 < v_1 < v_2 < \cdots < v_n < v_{n+1} = 1. \tag{3.35}$$

If the infinite triangular array $E(\mathbf{U})$ has nth row $E_n(U_n)$, and the infinite triangular array $E(\mathbf{V})$ has nth row $E_n(V_n)$, then

$$\Lambda_{n+1}(\mathbf{U}) = 0(\log(n+1))$$

and

$$\Lambda_{n+1}(\mathbf{V}) = 0(\log(n+1)).$$

Corollary 2 follows immediately from Theorem 1 with the appropriate choices of a_n in (3.3). The rational functions U_n and V_n play significant roles in the next section, and are further analyzed in [8].

4. A Class of Non-Rational Functions

The main objective of the present section is to prove, for every element f in a certain class $\mathbf{F}$ of functions, that

$$\Lambda_{n+1}(E_f) = 0(\log(n+1)), \tag{4.1}$$

where E_f is given in (2.6).

Definition 1. Let $\mathbf{F}$ be the set of all functions $f \in C^{\infty}(I)$ satisfying

$$\text{(a)} \quad f^{(n+1)}(x) \neq 0 \qquad \text{on } I,$$

and

$$\text{(b)} \quad \frac{1}{\alpha} \leqslant \left| \frac{f^{(n+2)}(x)}{f^{(n+1)}(x)} \right| \leqslant \frac{1}{\beta} \qquad \text{on } I, \tag{4.2}$$

for all n sufficiently large, where $\alpha \geqslant \beta > 0$ are constants possibly depending of f but not on n.

We observe that $f_\delta(x) = e^{\delta x}$, $\delta \neq 0$, is an element of $\mathbf{F}$. Strong unicity constants for functions $f \in \mathbf{F}$ are analyzed in [8], and a number of properties of $\mathbf{F}$ are itemized in that reference. Several lemmas that aid in proving (4.1) now precede the proof of the main theorem of Section 4.

Lemma 3. *Let $f \in \mathbf{F}$ with $f^{(n+1)}(x) \cdot f^{(n+2)}(x) > 0$ on I. If*

$$E_n(f) = \{x_0, x_1, ..., x_n, x_{n+1}\}, \tag{4.3}$$

where

$$-1 = x_0 < x_1 < \cdots < x_n < x_{n+1} = 1, \tag{4.4}$$

then

$$z_k < u_k < x_k < v_k < \zeta_k, \qquad k = 1, 2,..., n, \tag{4.5}$$

where $E_n(U_n) = \{u_l\}_{l=0}^{n+1}$ *and* $E_n(V_n) = \{v_l\}_{l=0}^{n+1}$ *are as in Corollary* 2, *and where* $\{z_l\}_{l=0}^{n+1}$ *and* $\{\zeta_l\}_{l=0}^{n}$ *are given in* (3.12).

Lemma 3 is proven in [8].

LEMMA 4. *Let* U_n *and* V_n *be the rational functions of Corollary* 2, *with extreme sets* $E_n(U_n) = \{u_l\}_{l=0}^{n+1}$ *and* $E_n(V_n) = \{v_l\}_{l=0}^{n+1}$. *Then*

$$v_k - u_k \leqslant C/n^2(1 - \xi_k^2), \qquad k = 1, 2,..., n, \tag{4.6}$$

where C is a positive constant not depending on n, and where

$$u_k < \xi_k < v_k.$$

Inequality (4.6) essentially follows from [8, (2.24)]. Lemma 4 implies that

$$\max_{1 \leqslant k \leqslant n} |v_k - u_k| = 0(1/n^2). \tag{4.7}$$

This is to be contrasted with

$$\max_{1 \leqslant k \leqslant n} |\zeta_k - z_k| = 0(1/n). \tag{4.8}$$

The additional sharpness displayed in (4.7) as contrasted to (4.8) will be subsequently exploited.

LEMMA 5. *Let* $\{u_l\}_{l=0}^{n+1}$ *and* $\{x_l\}_{l=0}^{n+1}$ *be the extreme sets given in* (3.35) *and* (4.4), *respectively. Then*

$$\prod_{\substack{j=0 \\ j \neq i}}^{n+1} |u_i - u_j| \leqslant C \prod_{\substack{j=0 \\ j \neq i}}^{n+1} |x_i - x_j|, \qquad i = 0, 1,..., n+1, \tag{4.9}$$

where C is positive and independent of n.

Proof. From [8, Lemma 3],

$$\begin{aligned} |u_i - u_j| &\leqslant |u_i - x_i| + |x_i - x_j| + |x_j - u_j| \\ &\leqslant 2A/n\,|x_i - x_j| + |x_i - x_j| \\ &= (1 + 2A/n)\,|x_i - x_j|, \qquad i = 0, 1,..., n+1, i \neq j. \end{aligned}$$

where A is a positive constant not depending on n. Since $\sup_n |1 + 2A/n|^n < +\infty$, (4.9) follows.

LEMMA 6. *Let $\{u_l\}_{l=0}^{n+1}$, $\{v_l\}_{l=0}^{n+1}$ and $\{x_l\}_{l=0}^{n+1}$ be the extreme points given in Lemma 3. For $x \in I$, select k such that $x_k \leqslant x \leqslant x_{k+1}$. Then*

$$\prod_{j=0}^{n+1} |x_j - x| \leqslant C \left\{ 1/n^2(1-\xi_{k+1}^2) \prod_{\substack{j=0\\ j \neq k+1}}^{n+1} |x - u_j| + \prod_{j=0}^{n+1} |x - u_j| \right\}, \quad (4.10)$$

where $u_{k+1} \leqslant \xi_{k+1} \leqslant v_{k+1}$, $k = 0,..., n$, and where C is a positive constant not depending on n.

Proof. From (4.5), for $k \geqslant 0$,

$$\prod_{j=0}^{k} |x - x_j| \leqslant \prod_{j=0}^{k} |x - u_j|. \quad (4.11)$$

Thus if $k = n$, (4.11) and the fact that $|x - u_{n+1}| = |x - x_{n+1}|$ combine to imply (4.10). Therefore we may assume that $k \leqslant n - 1$. Thus $k + 2 \leqslant j \leqslant n + 1$. Then

$$|x - x_j| = |x - u_j| \left| 1 + \frac{x_j - u_j}{u_j - x} \right|. \quad (4.12)$$

Now (4.5) implies that

$$\frac{x_j - u_j}{u_j - x} \leqslant \frac{v_j - u_j}{z_j - \zeta_{k+1}}, \qquad k + 2 \leqslant j \leqslant n + 1. \quad (4.13)$$

Therefore it follows from [8, Theorem 5, (2.19)] that

$$\sum_{j=k+2}^{n+1} \frac{v_j - u_j}{z_j - \zeta_{k+1}} \leqslant M < +\infty, \quad (4.14)$$

where M is a positive constant not depending on n. Hence (4.12), (4.13), and (4.14) imply that

$$\begin{aligned}\prod_{j=k+2}^{n+1} |x - x_j| &\leqslant \prod_{j=k+2}^{n+1} |x - u_j| \prod_{j=k+2}^{n+1} \left(1 + \frac{v_j - u_j}{z_j - \zeta_{k+1}}\right) \\ &\leqslant \prod_{j=k+2}^{n+1} |x - u_j| \exp\left[\sum_{j=k+2}^{n+1} \frac{v_j - u_j}{z_j - \zeta_{k+1}}\right] \\ &\leqslant \exp(M) \prod_{j=k+2}^{n+1} |x - u_j|. \end{aligned} \quad (4.15)$$

For $0 \leqslant k \leqslant n-1$, (4.11) and (4.15) imply that

$$\prod_{j=0}^{n+1} |x-x_j| \leqslant |x_{k+1}-x| \, |\exp(M)| \prod_{\substack{j=0 \\ j \neq k+1}}^{n+1} |x-u_j|$$

$$\leqslant \hat{M} \left| (x_{k+1}-u_{k+1}) \prod_{\substack{j=0 \\ j \neq k+1}}^{n+1} (|x-u_j|) + \prod_{j=0}^{n+1} (|x-u_j|) \right|,$$

where $\hat{M} = \exp(M)$. This last inequality, (4.5), and (4.6) now imply (4.10). ∎

LEMMA 7. *Let $\{x_l\}_{l=0}^{n+2}$ be the extreme points given in Lemma 3. Then*

$$2^n \prod_{\substack{j=0 \\ j \neq i}}^{n+1} |x-x_j| = 0(n). \tag{4.16}$$

The proof of this lemma follows from the proof of [8, Theorem 8].

LEMMA 8. *Suppose that $\{x_l\}_{l=0}^{n+1}$ are the extreme points given in Lemma 3. Let $x_k \leqslant x \leqslant k_{k+1}$, $k = 0,..., n$. Then*

$$\sum_{j=1}^{k-2} 1/(x-x_j) \leqslant C_1(n+1)^2 \log(n+1), \qquad k = 3,..., n. \tag{4.17}$$

and

$$\sum_{j=k+2}^{n} 1/(x_j-x) \leqslant C_2(n+1)^2 \log(n+1), \qquad k = 0,..., n-2, \tag{4.18}$$

where C_1 and C_2 are positive constants not depending on n.

Proof. We prove only (4.17); the proof of (4.18) is similar. For $1 \leqslant j \leqslant k-2$,

$$\begin{aligned} x-x_j &\geqslant z_k - z_{j+1} \\ &= \cos\left(\frac{n+1-k}{n+1}\right)\pi - \cos\left(\frac{n-j}{n+1}\right)\pi \\ &= \frac{k-(j+1)}{n+1}\pi \sin w_j, \end{aligned}$$

where w_j is between $(n+1-k)/(n+1)\pi$ and $(n-j)/(n+1)\pi$. Since $1 \leqslant j \leqslant k-2 \leqslant n-2$,

$$\sin \frac{w_j}{n+1} \geqslant \frac{M}{n+1},$$

where $M > 0$ is a positive constant not depending on n. Therefore

$$\begin{aligned}\sum_{j=1}^{k-2} \frac{1}{x-x_j} &\leqslant \sum_{j=1}^{k-2} \frac{1}{z_k - z_{j+1}} \\ &\leqslant \frac{(n+1)^2}{M\pi} \sum_{j=1}^{k-2} \frac{1}{k-(j+1)} \\ &\leqslant C_1 (n+1)^2 \log(n+1). \quad \blacksquare\end{aligned}$$

We are finally in a position to prove the main theorem of the present section and the principal theorem of the paper.

THEOREM 2. *Let* **F** *be the class of functions given by Definition* 1, *and let* E_f *be the infinite triangular array of nodes whose nth row is*

$$E_n(f) = \{x_0, x_1, ..., x_n, x_{n+1}\},$$

the extreme points of $e_n(f)$. *Then*

$$\Lambda_{n+1}(E_f) = 0(\log(n+1)). \tag{4.19}$$

Proof. First assume that $f \in \mathbf{F}$ satisfies $f^{(n+1)}(x) \cdot f^{(n+2)}(x) > 0$, $x \in I$. Let $U_n(x)$ be given by (3.34), and $\{u_l\}_{l=0}^{n+1}$ by (3.35). We assume that $E_n(f)$ has ordering (4.4), and that

$$x \neq x_l, \qquad l = 0, 1, ..., n+1.$$

Let $w_n(x)$, $x \in I$, be given by (3.13) with $a_n = \alpha(n+2)+2$. Then Lemma 5 and equality (2.29) in [6] imply that

$$\begin{aligned}\prod_{\substack{j=0\\ j\neq i}}^{n+1} \left|\frac{x-x_j}{x_i-x_j}\right| &\leqslant C \prod_{\substack{j=0\\ j\neq i}}^{n+1} \left|\frac{x-x_j}{u_i-u_j}\right| \\ &= \frac{C}{|w_n'(u_i)|} \prod_{\substack{j=0\\ j\neq i}}^{n+1} |x-x_j| \\ &\leqslant \frac{C \cdot n2^{n-1}[(a_n^2-1)^{1/2}+a_n]}{n[(a_n-1)\,n+(a_n-1)^{1/2}]} \prod_{\substack{j=0\\ j\neq i}}^{n+1} |x-x_j|. \end{aligned} \tag{4.20}$$

Applying (4.16) to (4.20) establishes that

$$\prod_{\substack{j=0\\ j\neq i}}^{n+1} \left|\frac{x-x_j}{x_i-x_j}\right| \leqslant C_2, \tag{4.21}$$

where C_2 does not depend on n. From (1.4)

$$\lambda_{n+1}(E_f, x) = \sum_{i=0}^{n+1} \prod_{\substack{j=0\\ j\neq i}}^{n+1} \left|\frac{x-x_j}{x_i-x_j}\right|. \tag{4.22}$$

Suppose that $x \in [x_k, x_{k+1}]$, where $0 \leqslant k \leqslant n$. Then (4.21) and (4.22) imply that

$$\begin{aligned}\lambda_{n+1}(E_f, x) \leqslant 5C_2 &+ \sum_{i=1}^{k-2} \prod_{\substack{j=0\\ j\neq i}}^{n+1} \left|\frac{x-x_j}{x_i-x_j}\right| \\ &+ \sum_{i=k+2}^{n} \prod_{\substack{j=0\\ j\neq i}}^{n+1} \left|\frac{x-x_j}{x_i-x_j}\right|,\end{aligned} \tag{4.23}$$

where again $\sum_{i=r}^{s} \theta_i = 0$ if $r > s$. Let

$$I_1 = \sum_{i=1}^{k-2} \prod_{\substack{j=0\\ j\neq i}}^{n+1} \left|\frac{x-x_j}{x_i-x_j}\right|, \tag{4.24}$$

and

$$I_2 = \sum_{i=k+2}^{n} \prod_{\substack{j=0\\ j\neq i}}^{n+1} \left|\frac{x-x_j}{x_i-x_j}\right|. \tag{4.25}$$

Again from Lemma 5,

$$\begin{aligned}I_1 &\leqslant \prod_{j=0}^{n+1} |x-x_j| \sum_{i=1}^{k-2} \frac{1}{|x-x_i| \prod_{j=0, j\neq i}^{n+1} |x_i-x_j|} \\ &\leqslant C \prod_{j=0}^{n+1} |x-x_j| \sum_{i=1}^{k-2} \frac{1}{|x-x_i| \prod_{j=0, j\neq i}^{n+1} |u_i-u_j|} \\ &= C \prod_{j=0}^{n+1} |x-x_j| \sum_{i=1}^{k-2} \frac{1}{|x-x_i|\, |w_n'(u_i)|}.\end{aligned}$$

An application of Lemma 6 to this inequality results in

$$I_1 \leqslant \bar{C}\left[\sum_{i=1}^{k-2} \frac{1}{|x-x_i|\,|w_n'(u_i)|}\right]$$

$$\cdot \left\{\frac{1}{n^2}(1-\xi_{k+1}^2) \prod_{\substack{j=0 \\ j\neq k+1}}^{n+1} |x-u_j| + \prod_{j=0}^{n+1} |x-u_j|\right\}$$

$$\leqslant \bar{C}\left[\sum_{i=1}^{k-2} \frac{1}{|x-x_i|\,|w_n'(u_i)|}\right]$$

$$\cdot \left\{\frac{1}{n^2}(1-\xi_{k+1}^2)\,\|w_n'\| + |w_n(x)|\right\}$$

$$= \bar{C}\left\{\frac{1}{n^2}(1-\xi_{k+1}^2) \sum_{i=1}^{k-2} \frac{1}{|x-x_i|} \cdot \frac{\|w_n'\|}{|w_n'(u_i)|}\right.$$

$$\left. + |w_n(x)| \sum_{i=1}^{k-2} \frac{1}{|x-x_i|\,|w_n'(u_i)|}\right\}.$$

Thus (3.14) and (4.5) yield

$$I_1 \leqslant \bar{C}\left\{C_2 \frac{1}{n^2}(1-\xi_{k+1}^2) \sum_{i=1}^{k-2} \frac{1}{|x-x_i|}\right.$$

$$\left. + |w_n(x)| \sum_{i=1}^{k-2} \frac{1}{|x-u_{i+1}|\,|w_n'(u_i)|}\right\}.$$

Inequalities (4.17) and (3.15) now imply that

$$I_1 \leqslant \hat{C}\left\{\log(n+1) + C_4\,|w_n(x)| \sum_{i=1}^{k-2} \frac{1}{|x-u_{i+1}|\,|w_n'(u_{i+1})|}\right\}.$$

Now Corollary 2 implies that

$$I_2 = 0(\log(n+1)).$$

By using a similar argument (e.g., (4.18)), one can show that

$$I_2 = 0(\log(n+1)).$$

Therefore if $f^{(n+1)}(x) \cdot f^{(n+2)}(x) > 0$, $x \in I$, then

$$\lambda_{n+1}(E_f, x) = 0(\log(n+1)), \qquad x \in I.$$

Now equality (1.5) implies conclusion (4.19).

To complete the proof of Theorem 2, assume that $f^{(n+1)}(x) \cdot f^{(n+2)}(x) < 0$, $x \in I$. By replacing f by $(-f)$ if necessary, we may assume that $f^{(n+1)}(x) > 0$. Define h by $h(x) = (-1)^{n+1} f(-x)$. Clearly $h^{(n+1)}(x) > 0$, $h^{(n+2)}(x) > 0$, $x \in I$, and $h \in \mathbf{F}$. Therefore the first part of the proof establishes that

$$\Lambda_{n+1}(E_h) = 0(\log(n+1)).$$

Let $-1 = \tau_0 < \tau_1 < \cdots < \tau_n < \tau_{n+1} = 1$ be the extreme points of $e_n(h)$. If $-1 = x_0 < x_1 < \cdots < x_n < x_{n+1} = 1$ are the extreme points of $e_n(f)$, then $\tau_i = -x_{n+1-i}$, $i = 0, 1, \ldots, n+1$. This observation ensures that

$$\Lambda_{n+1}(E_h) = \Lambda_{n+1}(E_f),$$

completing the proof.

5. Conclusion

In the preceding sections Lebesgue constants for certain infinite triangular arrays of nodes are examined.

It is shown for a certain class of rational functions $\mathbf{R} = \{r_n\}_{n=1}^{\infty}$ that the corresponding infinite triangular array of nodes whose nth row consists of the $n+2$ extreme points of $e_n(r_n)$ yields an asymptotically optimal Lebesgue constant.

This result is subsequently used to prove for every function f in a certain class of non-rational functions $\mathbf{F}$ that the Lebesgue constant constructed from the infinite triangular array of nodes whose nth row consists of the $n+2$ extreme points of $e_n(f)$ is also asymptotically optimal. In particular, the infinite triangular array of nodes whose nth row consists of the extreme points of the error function for $e^{\delta x}$, $\delta \neq 0$, yields a Lebesgue constant of order $\log(n+1)$.

References

1. L. Brutman, On the Lebesgue function for polynomial interpolation, *SIAM J. Numer. Anal.* **15** (1978), 694–704.
2. E. W. Cheney, "Introduction to Approximation Theory," McGraw-Hill, New York, 1966.
3. C. de Boor and A. Pinkus, Proof of conjectures of Bernstein and Erdos concerning the optimal nodes for polynomial interpolation, *J. Approx. Theory* **24** (1978), 289–303.
4. H. Ehlich and K. Zeller, Auswertung der Normen von Interpolations-operatoren, *Math. Ann.* **164** (1966), 105–112.

5. P. ERDOS, Problems and results on the theory of interpolation, II, *Acta Math. Acad. Sci. Hungar.* **12** (1961), 235–244.
6. M. S. HENRY AND L. R. HUFF, On the behavior of the strong unicity constant for changing dimension, *J. Approx. Theory* **22** (1978), 85–94.
7. M. S. HENRY, J. J. SWETITS, AND S. E. WEINSTEIN, Lebesgue and strong uncity constants, *in* "Approximation III" (E. W. Cheney, Ed.), pp. 507–512, Academic Press, New York, 1980.
8. M. S. HENRY, J. J. SWETITS, AND S. E. WEINSTEIN, On extremal sets and strong unicity constants for certain C^{∞} functions, *J. Approx. Theory* **37** (1983), 155–174.
9. T. KILGORE, A characterization of the Lagrange interpolating projection with minimal Tchebycheff norm, *J. Approx. Theory* **24** (1978), 273–288.
10. G. MEINARDUS, "Approximation of Functions, Theory and Numerical Methods," Springer-Verlag, New York/Berlin, 1967.
11. T. J. RIVLIN, "An Introduction to the Approximation of Functions," Ginn (Blaisdell), Boston, 1969.
12. T. J. RIVLIN, "The Chebyshev Polynomials," Wiley–Interscience, New York, 1974.
13. J. H. ROWLAND, On the location of the deviation points in Chebyshev approximation by polynomials, *SIAM J. Numer. Anal.* **6** (1969), 118–126.
14. G. SZEGÖ, "Orthogonal Polynomials," American Mathematical Society, Providence, R.I., 1959.